5.6.1 Gagliardo-Nirenberg-Sobolev inequality

Toolbox

(i) **(Hölder's inequality)** Assume $1 \le p, q \le \infty, \dfrac{1}{p} + \dfrac{1}{q} = 1$. Then if $u \in L^p(U), v \in L^q(U)$, we have

$$\int_U |uv|\, dx \le \|u\|_{L^p(U)} \|v\|_{L^q(U)} \tag{1}$$

Proof: Use Young's inequality.

(ii) If U is bounded, $f \in L^{p'}(U)$ for $1 \le r \le p'$, then there exists $C = C(U)$, such that

$$\|f\|_{L^r(U)} \le C \|f\|_{L^{p'}(U)} \tag{2}$$

Proof: In Hölder's inequality, choose $u = |f|^r$, $v = 1$, and $p = \dfrac{p'}{r}$

(iii) **(General Hölder's inequality)** Let $1 \le p_1, \cdots p_m \le \infty$ with $\sum_{k=1}^{m} \dfrac{1}{p_k} = 1$, and assume $u_k \in L^{p_k}(U)$ for $k = 1, \cdots, m$ Then

$$\int_U |u_1 \cdots u_m|\, dx \le \sum_{k=1}^{m} \|u_k\|_{L^{p_k}(U)} \tag{3}$$

Proof: By induction.

Theorem 1 (Gagliardo-Nirenberg-Sobolev inequality) *Assume* $\underline{1 \le p < n}$. *There exists a constant* C*, depending only on* p *and* n*, such that*

$$\|u\|_{L^{p*}(\mathbb{R}^n)} \le C \|Du\|_{L^p(\mathbb{R}^n)} \tag{4}$$

for all $u \in C_c^1(\mathbb{R}^n)$*, where* $p* := \dfrac{np}{n-p}$ *is called the* ***Sobolev conjugate*** *of* p*.*

Motivation: Why do we define the **Sobolev conjugate** to be $p* := \dfrac{np}{n-p}$?
Assume $\forall u \in C_c^1(\mathbb{R}^n), u \not\equiv 0$, we have the following inequality for some fixed but unkonwn q:

$$\|u\|_{L^q(\mathbb{R}^n)} \le C \|Du\|_{L^p(\mathbb{R}^n)} \tag{5}$$

where $C = C(n,p)$ For $\lambda > 0$, we define the rescaled function:

$$u_\lambda(x) := u(\lambda x),$$

which is still a $C_c^1(\mathbb{R}^n)$ function. Therefore by applying (5) to $u_\lambda(x)$ we have

$$\|u_\lambda\|_{L^q(\mathbb{R}^n)} \le C \|Du_\lambda\|_{L^p(\mathbb{R}^n)} \tag{6}$$

Now, by change of variables we have

$$\begin{aligned} \|u_\lambda\|_{L^q(\mathbb{R}^n)} &= \lambda^{-\frac{n}{q}} \|u\|_{L^q(\mathbb{R}^n)} \\ \|Du_\lambda\|_{L^p(\mathbb{R}^n)} &= \lambda^{1-\frac{n}{p}} \|u\|_{L^p(\mathbb{R}^n)} \end{aligned} \tag{7}$$

Plug (7) in (6) we have

$$\|u\|_{L^q(\mathbb{R}^n)} \le C\lambda^{1-\frac{n}{p}+\frac{n}{q}} \|Du\|_{L^p(\mathbb{R}^n)}$$

If $1 - \frac{n}{p} + \frac{n}{q} \neq 0$, we can let

$$\begin{cases} \lambda \to 0, & 1 - \dfrac{n}{p} + \dfrac{n}{q} > 0 \\ \lambda \to \infty, & 1 - \dfrac{n}{p} + \dfrac{n}{q} < 0 \end{cases},$$

which leads to the contradiction that $\|u\|_{L^q(\mathbb{R}^n)} = 0$. Thus, to have an inequality of the the same form as (5), we must have

$$1 - \frac{n}{p} + \frac{n}{q} = 0,$$

that is,

$$\frac{1}{q} = \frac{1}{p} - \frac{1}{n}, \quad \text{or} \quad q = \frac{np}{n-p} \tag{8}$$

Proof to Gagliardo-Nirenberg-Sobolev inequality:

Step 1. Assume $p = 1$, then $p* = \frac{n}{n-1}$

Since u has compact support in $\mathbb{R}^n$, for each $i = 1, 2, \cdots, n$ and $x \in \mathbb{R}^n$. we have

$$u(x) = \int_{-\infty}^{x_i} u_{x_i}(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n) dy_i;$$

and so

$$\begin{aligned} |u(x)| &\le \int_{-\infty}^{x_i} |u_{x_i}(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n)|\, dy_i \\ &\le \int_{-\infty}^{x_i} |Du(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n)|\, dy_i \\ &\le \int_{-\infty}^{\infty} |Du(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n)|\, dy_i \quad (i = 1, 2, \cdots, n) \end{aligned}$$

Consequently,

$$|u(x)|^{\frac{n}{n-1}} \le \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} |Du(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n)|\, dy_i \right)^{\frac{1}{n-1}} \tag{9}$$

Integrate this inequality with respect to x_1:

$$\begin{aligned} \int_{-\infty}^{\infty} |u|^{\frac{n}{n-1}}\, dx_1 &\le \int_{-\infty}^{\infty} \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} |Du|\, dy_i \right)^{\frac{1}{n-1}} dx_1 \\ &= \left(\int_{-\infty}^{\infty} |Du|\, dy_1 \right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} |Du|\, dy_i \right)^{\frac{1}{n-1}} dx_1 \\ &\le \left(\int_{-\infty}^{\infty} |Du|\, dy_1 \right)^{\frac{1}{n-1}} \left(\prod_{i=2}^{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du|\, dx_1 dy_i \right)^{\frac{1}{n-1}} \end{aligned} \tag{10}$$

the last inequality resulting from the general Hölder's inequality (3).
Now integrate (10) with respect to x_2:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |u|^{\frac{n}{n-1}}\, dx_1 dx_2 \le \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du|\, dx_1 dy_2 \right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{\substack{i=1 \\ i \ne 2}}^{n} I_i^{\frac{1}{n-1}} dx_2$$

for

$$I_1 := \int_{-\infty}^{\infty} |Du|\, dy_1, \quad I_i := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du|\, dx_1 dy_i \quad (i = 3, \cdots, n)$$

Applying the general Hölder's inequality again, we find

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |u|^{\frac{n}{n-1}}\, dx_1 dx_2 \le \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du|\, dx_1 dy_2 \right)^{\frac{1}{n-1}} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du|\, dy_1 dx_2 \right)^{\frac{1}{n-1}} \\ \prod_{i=3}^{n} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du|\, dx_1 dx_2 dy_i \right)^{\frac{1}{n-1}} \end{aligned}$$

We continue by integrating with respect to $x_3, \cdots x_n$, eventually to find

$$\begin{aligned}\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}}\,dx &\le \prod_{i=1}^{n}\left(\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|Du|\,dx_1\cdots dy_i\cdots dx_n\right)^{\frac{1}{n-1}} \\ &= \left(\int_{\mathbb{R}^n}|Du|^{\frac{n}{n-1}}\,dx\right)^{\frac{n}{n-1}}.\end{aligned} \tag{11}$$

This is estimate (4) for $p = 1$

Step 2. Consider now the case that $1 < p < n$. We apply estimate (11) to $v := |u|^{\gamma}$, where $\gamma > 1$ is to be selected. Then

$$\begin{aligned}\left(\int_{\mathbb{R}^n}|u|^{\frac{\gamma n}{n-1}}\,dx\right)^{\frac{n-1}{n}} &\le \int_{\mathbb{R}^n}\left|D\,|u|^{\gamma}\right|dx = \gamma\int_{\mathbb{R}^n}|u|^{\gamma-1}\,|Du|\,dx \\ &\le \gamma\left(\int_{\mathbb{R}^n}|u|^{(\gamma-1)\frac{p}{p-1}}\,dx\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^n}|Du|^{p}\,dx\right)^{\frac{1}{p}}\end{aligned} \tag{12}$$

If we choose γ so that $\frac{\gamma n}{n-1} = (\gamma-1)\frac{p}{p-1}$, that is, we set

$$\gamma := \frac{p(n-1)}{n-p} > 1,$$

And it turns out

$$\frac{\gamma n}{n-1} = (\gamma-1)\frac{p}{p-1} = \frac{np}{n-1} = p*$$

So estimate (12) becomes

$$\|u\|_{L^{p*}(\mathbb{R}^n)} \le \frac{p(n-1)}{n-p}\,\|Du\|_{L^{p}(\mathbb{R}^n)}\,,$$

and constant in (4) is $C = \frac{p(n-1)}{n-p}$ □

Theorem 2 (Estimate for $W^{1,p}, 1 \le p < n$). Let U be a bounded, open subset of $\mathbb{R}^n$, and suppose ∂U is C^1. Assume $1 \le p < n$, and $u \in W^{1,p}(U)$. Then $u \in L^{p*}(U)$, with the estimate

$$\|u\|_{L^{p*}(U)} \le C\,\|u\|_{W^{1,p}(U)} \tag{13}$$

the constant C depending only on p, n and U.

Proof. Since ∂U is C^1 and $u \in W^{1,p}(U)$, from theorem 1 of section 5.4, there exists an extension $Eu = \bar{u} \in W^{1,p}(\mathbb{R}^n)$, such that

$$\begin{cases}\bar{u} = u \text{ in } U, \bar{u} \text{ has compact support in } \mathbb{R}^n, \text{ and} \\ \|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} \le C_e\,\|u\|_{W^{1,p}(U)}\end{cases} \tag{14}$$

Because $\bar{u}$ has compact support, from theorem 1 of section 5.3, there exist functions $u_m \in C_c^{\infty}(\mathbb{R}^n)(m = 1, 2, \cdots)$ such that

$$u_m \to \bar{u} \quad \text{in } W^{1,p}(\mathbb{R}^n) \tag{15}$$

And now according to Theorem 1 we just proved, $\|u_m - u_l\|_{L^{p*}(\mathbb{R}^n)} \le C\,\|Du_m - Du_l\|_{L^p(\mathbb{R}^n)}$ for all $m, l \ge 1$. Thus

$$u_m \to \bar{u} \quad \text{in } L^{p*}(\mathbb{R}^n) \tag{16}$$

as well. Since Theorem 1 in this section also implies $\|u_m\|_{L^{p*}(\mathbb{R}^n)} \le C\,\|Du_m\|_{L^p(\mathbb{R}^n)}$, (15) and (16) yield the bound

$$\|\bar{u}\|_{L^{p*}(\mathbb{R}^n)} \le C\,\|D\bar{u}\|_{L^p(\mathbb{R}^n)}$$

and

$$\|u\|_{L^{p*}(U)} = \|\bar{u}\|_{L^{p*}(U)} \leq \|\bar{u}\|_{L^{p*}(\mathbb{R}^n)} \leq C_1 \|D\bar{u}\|_{L^p(\mathbb{R}^n)} \leq \|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq C_1 C_e \|u\|_{W^{1,p}(U)} = C \|u\|_{W^{1,p}(U)}$$

□

Remark: If ∂U is Lipschitz but not C^1, then the theorem is not true. (See pp.318-319, *A First Course in Sobolev Spaces* by Giovanni Leoni)

Corollary (Sobolev embedding for $W^{k,p}$). If $u \in W^{k,p}(U)$ for some bounded, open subset of $\mathbb{R}^n$, and $1 \leq p < \frac{n}{k}$, then

$$\|u\|_{L^{\frac{np}{n-kp}}(U)} \leq C \|u\|_{W^{k,p}(U)} \tag{17}$$

Proof: I will illustrate the idea using $k = 2$. For complete proof please use induction. Since $u \in W^{2,p}(U)$, $u, Du \in W^{1,p}(U) \subset L^p(U)$. And we have the estimates (13) for both u and Du, i.e.

$$\|u\|_{L^{\frac{np}{n-p}}(U)} \leq C_1 \|u\|_{W^{1,p}(U)}$$
$$\|Du\|_{L^{\frac{np}{n-p}}(U)} \leq C_2 \|u\|_{W^{1,p}(U)}$$

Thus by definition of Sobolev spaces, $u \in W^{1,\frac{np}{n-p}}(U)$. Now if we use (13) again for u we will have the estimate

$$\|u\|_{L^{\frac{np}{n-2p}}(U)} \leq C \|u\|_{W^{2,p}(U)}$$

Remark: This implies that $W^{k,p}(U) \subset L^{\frac{np}{n-kp}}(U)$.

□

Theorem 3 (Estimates for $W_0^{1,p}, 1 \leq p < n$). Assume U is a bounded open subset of $\mathbb{R}^n$. Suppose $u \in W_0^{1,p}(U)$ for some $1 \leq p < n$. Then we have the estimate

$$\|u\|_{L^q(U)} \leq C \|Du\|_{L^p(U)} \tag{18}$$

for all $q \in [1, p*]$, the constant $C = C(p, q, n, U)$.

Proof Since $u \in W_0^{1,p}(U)$, there exists functions $u_m \in C_c^\infty(U) \quad (m = 1, 2, \cdots)$ converging to u in $W^{1,p}(U)$. We extend each function u_m to be 0 on $\mathbb{R}^n - \bar{U}$ and using the arguments similar to the proof of Theorem 2 we will have

$$\|u\|_{L^{p*}(U)} \leq C \|u\|_{W^{1,p}(U)}$$

As $|U| < \infty$, by (2) we furthermore have

$$\|u\|_{L^q(U)} \leq C \|Du\|_{L^p(U)} \quad \forall q \in [1, p*]$$

□

Corollary (Classical Poincaré inequality) Let U be a bounded, open subset of $\mathbb{R}^n$. For all $1 \leq p \leq \infty$, if $u \in W_0^{1,p}(U)$, then we have the estimate

$$\|u\|_{L^p(U)} \leq C \|Du\|_{L^p(U)} \tag{19}$$

Proof: We will prove this for the cases $1 \leq p < n$, $n \leq p < \infty$ and $p = \infty$

(i) For $1 \leq p < n$, since $p < p*$, (19) is just a special case of (18)

(ii) For $n \leq p < \infty$, we can choose $1 \leq q < n$ such that $q < n \leq p < q*$ using the fact that $q* \to \infty$ as $q \to n$. Since $W_0^{1,p}(U) \subset W_0^{1,q}(U)$, applying (18) to $W_0^{1,q}(U)$ and (2) to $\|Du\|_{L^q(U)}$ we have the estimate

$$\|u\|_{L^p(U)} \leq C' \|Du\|_{L^q(U)} \leq C \|Du\|_{L^p(U)}$$

(iii) For $p = \infty$, using the fundamental theorem of calculus, we have

$$\begin{aligned} |u(x_1, x_2, \cdots, x_n)| &= \left| \int_{y_1}^{x_1} \partial_{x_1} f(t, x_2, \cdots, x_n) dt \right| \\ &\leq \int_{y_1}^{x_1} |Du(t, x_2, \cdots, x_n)| \, dt \\ &\leq \operatorname{diam}(U) \|Du\|_{L^\infty(U)} \end{aligned} \tag{20}$$

where $y_1 < x_1$ and y_1 is small enough such that $(y_1, y_2, \cdots, y_n)$ is outside of $\bar{U}$ for all $y_i \in \mathbb{R}$ $(i = 2, 3, \cdots, n)$. Now we take the supremum of the left hand side of (20) we have

$$\|u\|_{L^\infty(U)} \leq \operatorname{diam}(U) \|Du\|_{L^\infty(U)}$$

□

The borderline case $\mathbf{p = n}$. We next assume that

$$p = n$$

Because of the estimate (13)and the fact that $p \to n, p* = \frac{np}{n-p} to + \infty$, we might expect that $u \in L^\infty(U)$ provided $u \in W^{1,n}(U)$. *This is however false if $n > 1$.*

Example 1: Let $n > 1$, and $U = B^{\text{open}}(0, 1), u = \log\log(1 + \frac{1}{|x|})$. Then $u \in W^{1,n}(U)$ but $u \notin L^\infty(U)$.

Proof: Use spherical coordinates and some tricks in Calculus I.